

TOPICS IN COMPLEX ANALYSIS @ EPFL, FALL 2024

SOLUTION SKETCHES TO HOMEWORK 2

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Homework 2.1 (A convergence criterion). Let $D \subset \mathbb{C}$ be a domain and let $(f_n)_{n \in \mathbb{N}}$ be a locally uniformly bounded sequence of holomorphic functions $f_n: D \rightarrow \mathbb{C}$. Assume there exists $z_0 \in D$ such that for every $k \in \mathbb{N}_0$, the following limit exists:

$$\lim_{n \rightarrow \infty} f_n^{(k)}(z_0).$$

- a. With elementary methods from analysis, show the following “lemma”. Let $(z_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{C}$ and let $z \in \mathbb{C}$. Then $(z_n)_{n \in \mathbb{N}}$ converges to z if and only if every subsequence of $(z_n)_{n \in \mathbb{N}}$ has a further subsequence which converges to z ¹. Can one replace the second condition by “ $(z_n)_{n \in \mathbb{N}}$ has a subsequence converging to z ”?
- b. Use a. to show $(f_n)_{n \in \mathbb{N}}$ converges locally uniformly to some function $f: D \rightarrow \mathbb{C}$ ².

Solution. a. It is clear that if $(z_n)_{n \in \mathbb{N}}$ converges to z , then every subsequence (of every subsequence) of $(z_n)_{n \in \mathbb{N}}$ converges to z .

Conversely, assume $(z_n)_{n \in \mathbb{N}}$ does not converge to z . Then (by reversing the definition of convergence logically) there exists $\varepsilon > 0$ such that for every $N \in \mathbb{N}$ there exists $n_N \geq N$ such that $|z_{n_N} - z| > \varepsilon$. Up to removing possible multiple (but finite) appearances of the n_N ’s, is clear from this construction that no subsequence of $(z_n)_{n \in \mathbb{N}}$ converges to z .

One cannot replace the second condition by the claimed statement. This is easily seen by considering an alternating sequence which does not converge at all.

b. We first prove $(f_n)_{n \in \mathbb{N}}$ converges pointwise to a function f to be constructed. As seen in the proof of Vitali’s theorem from the lecture, the nonconvergence of $(f_n)_{n \in \mathbb{N}}$ at a point $z' \in D$ implies the existence of two holomorphic functions $h, g: D \rightarrow \mathbb{C}$ and two subsequences $(f_{n_{k,1}})_{k \in \mathbb{N}}$ and $(f_{n_{k,2}})_{k \in \mathbb{N}}$ such that

- $f_{n_{k,1}} \rightarrow h$ and $f_{n_{k,2}} \rightarrow g$ locally uniformly as $k \rightarrow \infty$ and
- $h(z') \neq g(z')$.

Applying Theorem 1.5 from the lecture notes, we infer for every $m \in \mathbb{N}$ that all derivatives of the two subsequences converge locally uniformly to the corresponding derivatives of h and g , respectively. Thus by analyticity of h and g , the set $\{h = g\}$ is non-empty and contains a small ball $B_r(z_0)$. By the identity theorem we conclude $h = g$, a contradiction.

We then define $f: D \rightarrow \mathbb{C}$ in the evident way, i.e. $f(z)$ is the limit of the sequence $(f_n(z))_{n \in \mathbb{N}}$ whose convergence we have just shown, where $z \in D$.

It remains to show $(f_n)_{n \in \mathbb{N}}$ converges locally uniformly to f . By Remark 1.2 from the lecture notes, this is equivalent to showing that for every compact subset $K \subset D$,

$$\lim_{n \rightarrow \infty} \sup_{z \in K} |f_n(z) - f(z)| = 0. \tag{2.1}$$

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¹This statement may look odd at first glance. It appears to make a simple statement about convergence more complicated. However, as you will hopefully acknowledge in solving b. (and in later courses), its usefulness throughout analysis comes in fact from the *backward* implication. In applications, one usually cannot show the convergence of an entire sequence directly, but compactness often allows one to extract subsequences one can handle better.

²**Hint.** Establish pointwise convergence first.

Let such a K be fixed. Consider an arbitrary subsequence $(f_{n_k})_{k \in \mathbb{N}}$. By Montel's theorem, a further subsequence $(f_{n_{k_j}})_{j \in \mathbb{N}}$ converges locally uniformly to a holomorphic function $g: D \rightarrow \mathbb{C}$. On the other hand, the extracted subsubsequence converges pointwise to f , which forces $g = f$ — in particular, f is holomorphic. The locally uniform convergence previously observed implies $\lim_{j \rightarrow \infty} a_{n_{k_j}} = 0$, where

$$a_n := \sup_{z \in K} |f_n(z) - f(z)|.$$

In other words, we have just shown every subsequence of $(a_n)_{n \in \mathbb{N}}$ contains a further subsequence converging to zero. By a., this implies $a_n \rightarrow 0$ as $n \rightarrow \infty$, which is (2.1).

Homework 2.2 (A compactness criterion). Let $B_1(0) \subset \mathbb{C}$ denote the open unit disk. Define the family of functions

$$\mathcal{F} = \left\{ f: B_1(0) \rightarrow \mathbb{C} : f \text{ holomorphic, } f(z) = \sum_{k=0}^{\infty} a_k z^k \text{ for every } z \in B_1(0), \right. \\ \left. |a_k| \leq 1 \text{ for every } k \in \mathbb{N}_0 \right\}.$$

Let $(f_n)_{n \in \mathbb{N}}$ be an arbitrary sequence in $\mathcal{F}$.

- Show $(f_n)_{n \in \mathbb{N}}$ subconverges locally uniformly to a function $f: B_1(0) \rightarrow \mathbb{C}$.
- Show f belongs to $\mathcal{F}$.

Solution. a. Since $|a_k| \leq 1$ for every $k \in \mathbb{N}_0$, the triangle inequality yields

$$|f(z)| \leq \sum_{k=0}^{\infty} |a_k| |z|^k \leq \sum_{k=0}^{\infty} |z|^k = \frac{1}{1 - |z|} \quad (2.2)$$

for every $f \in \mathcal{F}$ and every $z \in B_1(0)$. Given any compact set $K \subset B_1(0)$, by continuity the function $|\cdot|$ assumes its maximum on K . This maximum is clearly less than one. Combining this with (2.2) shows $(f_n)_{n \in \mathbb{N}}$ is locally uniformly bounded. Applying Montel's theorem yields the claim.

b. As f_n is holomorphic for every $n \in \mathbb{N}$, Theorem 1.3 from the lecture notes implies f is holomorphic. The claimed series representation holds for every holomorphic function on $B_1(0)$, hence for f . Thus it remains to show that the modulus of the coefficients in this series representation are no larger than 1. Denote the coefficients of f_n by a_k^n and the coefficients of f by a_k . Then we have $a_k^n = f_n^{(k)}(0)/k!$ for every $k \in \mathbb{N}_0$. From Theorem 1.5 in the lecture notes, up to passing to an appropriate subsequence we deduce $f_n^{(k)}(0) \rightarrow f^{(k)}(0)$ as $n \rightarrow \infty$. Continuity yields $|a_k| = |f^{(k)}(0)|/k! \leq 1/k! \leq 1$, which proves the claim.

Homework 2.3 (An extremal problem in the proof of the Riemann mapping theorem*). Let $D \subseteq \mathbb{C}$ be a simply connected domain containing zero. Show there exists a holomorphic function $f: D \rightarrow \mathbb{C}$ which attains the supremum

$$s_0 := \sup\{|f'(0)| : f: D \rightarrow B_1(0), f \text{ holomorphic and injective, } f(0) = 0\}.$$

We consider a sequence $(f_n)_{n \in \mathbb{N}}$ of functions $f_n: D \rightarrow B_1(0)$ which are admissible in the previous supremum³ such that $|f_n'(0)| \rightarrow s_0$ as $n \rightarrow \infty$.

- Show $(f_n)_{n \in \mathbb{N}}$ subconverges to a holomorphic function $f: D \rightarrow B_1(0)$.
- Show s_0 is a real number.
- Show the above supremum is attained for the function f from a. (in particular, f is admissible in the definition of s_0).

³The nonemptiness of the class of functions satisfying the desired properties is indeed not obvious. You may assume this fact as given without proof (which will be given later during the lecture, based on D being simply connected).

Homework 2.4 (Locally normal vs. locally uniform convergence). Let $(f_j)_{j \in \mathbb{N}}$ be a sequence of functions $f_j: U \rightarrow \mathbb{C}$, where $U \subset \mathbb{C}$ is open. The series $\sum_{j=1}^{\infty} f_j$ will be called *locally normally convergent* if for every $z_0 \in U$ there exists $r > 0$ such that

$$\sum_{j=1}^{\infty} \sup_{z \in B_r(z_0)} |f_j(z)| < \infty.$$

- Show if $\sum_{j=1}^{\infty} f_j$ is locally normally convergent, then this series is locally uniformly convergent.
- Give an example of a sequence $(f_j)_{j \in \mathbb{N}}$ of not necessarily holomorphic functions $f_j: \mathbb{C} \rightarrow \mathbb{C}$ which shows the converse of a. is in general false.

Solution. a. Local normal convergence implies in particular the series $\sum_{j=1}^{\infty} f_j(z)$ exists in $\mathbb{C}$ for every $z \in U$. Now fix $z_0 \in U$ and let $r > 0$ be given by the definition of local normal convergence. Then for any $z \in B_r(z_0)$, the triangle inequality implies

$$\sup_{z \in B_r(z_0)} \left| \sum_{j=1}^{\infty} f_j(z) - \sum_{j=1}^n f_j(z) \right| \leq \sup_{z \in B_r(z_0)} \sum_{j=n+1}^{\infty} |f_j(z)| \leq \sum_{j=n+1}^{\infty} \sup_{z \in B_r(z_0)} |f_j(z)|.$$

By summability, the right hand side of the above inequality converges to zero as $n \rightarrow \infty$. This proves the claim.

b) In the above proof there are two estimates which are not sharp; we employ these to produce a counterexample. While the first was about the triangle inequality, the second interchanged sum and suprema. We exploit the first and consider sign-changing functions. For $j \in \mathbb{N}$ define $f_j: \mathbb{C} \rightarrow \mathbb{C}$ by $f_j(z) := (-1)^{j+1} e^z / j$. Since the alternating harmonic series converges to $\log 2$ and the exponential function is bounded on every compact set $K \subset \mathbb{C}$, we see for every $z_0 \in \mathbb{C}$ that

$$\limsup_{n \rightarrow \infty} \sup_{z \in B_1(z_0)} \left| \log 2 e^z - \sum_{j=1}^n \frac{(-1)^{j+1}}{j} e^z \right| \leq \limsup_{n \rightarrow \infty} e^{|z_0|+1} \left| \log 2 - \sum_{j=1}^n \frac{(-1)^{j+1}}{j} \right| = 0.$$

Hence the sequence $(S_n)_{n \in \mathbb{N}}$ of partial sums $S_n = \sum_{j=1}^n f_j$ converges locally uniformly to the function $g: \mathbb{C} \rightarrow \mathbb{C}$ defined through $g(z) := \log 2 e^z$.

On the other hand, for $z_0 = 0$ we obtain for every $r > 0$ that

$$\sum_{j=1}^{\infty} \sup_{z \in B_r(0)} |f_j(z)| \geq \sum_{j=1}^{\infty} \frac{1}{j} = \infty,$$

which shows the series $\sum_{j=1}^{\infty} f_j$ is not locally normally convergent.